MATH 2020 A
2020/21

Lecture 7

- triple integral

Last time we established

<u>Theorem 1</u> Let F be a continuous function in B then

$$\iiint_B F = \iint_R \int_e^f F\, dz\, dA(x,y).$$

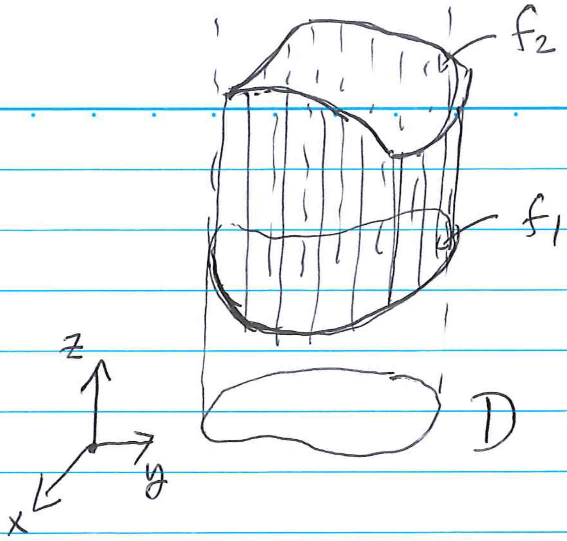
Here

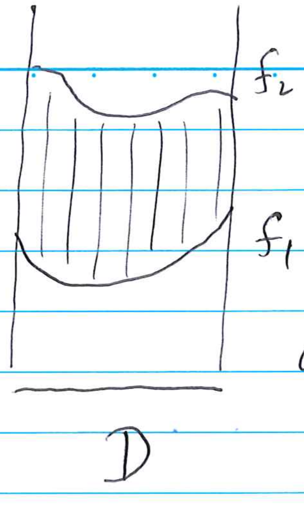
$$B = R \times [e,f]$$
$$\equiv [a,b] \times [c,d] \times [e,f] \quad \text{rectangular box.}$$

Next, we consider integrals over a region. We'll not discuss general domains but focus on a special type, namely, those of the form

$$\Omega = \{ (x,y,z) : (x,y) \in D,\ f_1(x,y) \leq z \leq f_2(x,y) \}$$

where $D \subseteq \mathbb{R}^2$ is a region.





a cross section

As usual we pick a rectangular box B to contain Ω and extend F to $\tilde{F}$ by setting it 0 outside Ω.

We define

$$\iiint_{\Omega} F \overset{\text{def}}{=} \iiint_{B} \tilde{F} .$$

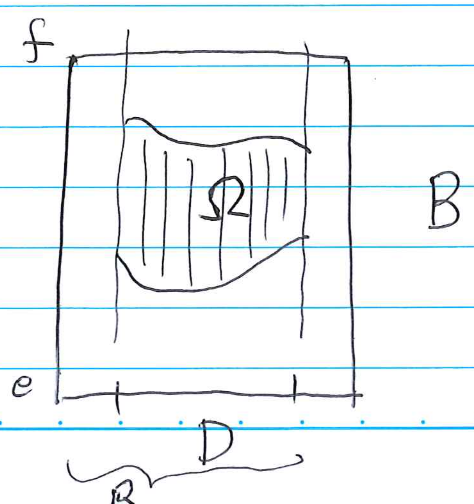
Whenever F is continuous in Ω, $\tilde{F}$ is integrable in B and Theorem 1 holds.

<u>Theorem 2</u> Let F be conti in Ω. Then

$$\iiint_{\Omega} F = \iint_{D} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dA(x,y).$$

Proof.

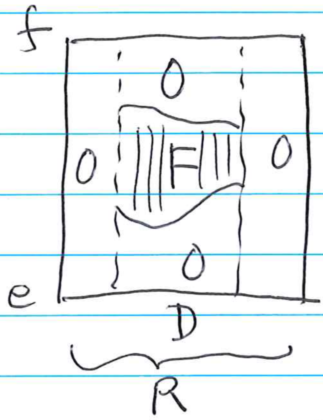
a cross section



By def.

$$\iiint_{\Omega} F = \iiint_{B} \tilde{F} \qquad B = R\times[e,f]$$

$$= \iint_{R} \int_{e}^{f} \tilde{F}$$



$$= \iint_{R\setminus D} \int_{e}^{f} \tilde{F} + \iint_{D} \int_{e}^{f} \tilde{F}$$

$$= 0 + \iint_{D} \int_{e}^{f} \tilde{F} \qquad (\because \tilde{F} = 0 \text{ on } R\setminus D)$$

$$= \iint_{D} \left(\int_{e}^{f_1(x,y)} \tilde{F} + \int_{f_1(x,y)}^{f_2(x,y)} \tilde{F} + \int_{f_2(x,y)}^{f} \tilde{F} \right)$$

$$= \iint_{D} \left(0 + \int_{f_1(x,y)}^{f_2(x,y)} \tilde{F} + 0 \right) \qquad (\because \tilde{F} = 0 \text{ there})$$

$$= \iint_{D} \int_{f_1(x,y)}^{f_2(x,y)} F \qquad (\because \tilde{F} = F \text{ in } \Omega)$$

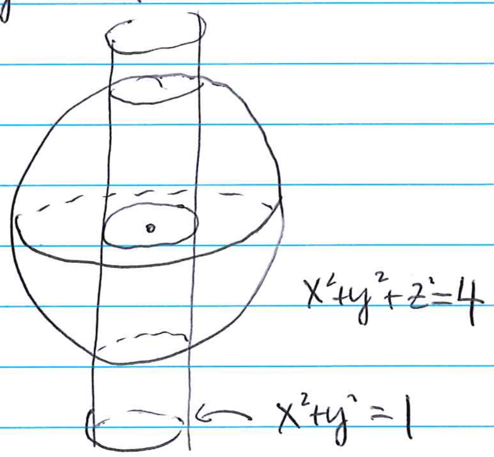
#

When $F=1$, from the def. of triple integral,

$$|\Omega| = \iiint_{\Omega} 1 \text{ is the volume of } \Omega.$$

e.g. 1. Find the volume of the region bounded by the sphere $x^2+y^2+z^2=4$ and cylinder $x^2+y^2=1$.

Just consider half of the region



$$\Omega = \{(x,y,z) : x^2+y^2 \le 1, \; 0 \le z \le f_2(x,y)\}$$

where $f_2(x,y) = \sqrt{4-x^2-y^2}$.

$\therefore$ The volume $= 2|\Omega|$

$$= 2\iint_D \int_0^{\sqrt{4-x^2-y^2}} 1 \, dz \, dA, \quad D = \{(x,y) : x^2+y^2 \le 1\}$$

$$= 2\iint_D \sqrt{4-x^2-y^2} \, dA$$

$$= 2\int_0^{2\pi}\int_0^1 \sqrt{4-r^2} \, r \, dr \, d\theta$$

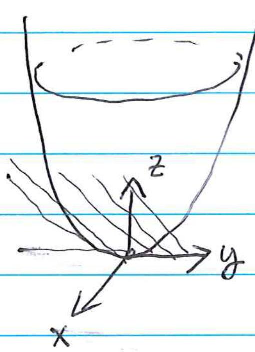
$$= 4\pi \times \frac{1}{2}(4-t)^{3/2}\Big|_1^0 = 2\pi(8-3^{3/2}) \quad \#$$

e.g. 2 Express the triple integral over Ω as iterated integral

Here Ω is the region bdd by $z = x^2+y^2$ and $z = x$.

$z = x^2+y^2$ is a paraboloid

and $z = x$ is a plane.



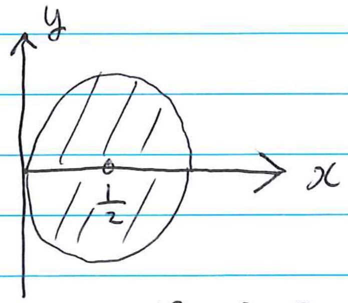
Their intersection (x,y,z) satisfies

$$x^2+y^2 = z = x$$

or $x^2+y^2-x = 0$

or $(x-\frac{1}{2})^2+y^2 = \frac{1}{4}$.

That's $D = \{(x,y) : (x-\frac{1}{2})^2+y^2 \le \frac{1}{4}\}$ a disk



$r = \cos\theta$

in polar coor

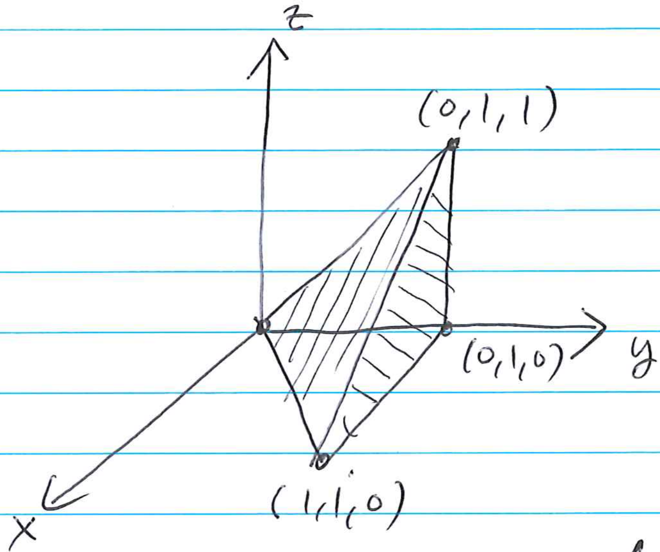
$$\therefore \iiint_{\Omega} F = \iint_{D} \int_{x^2+y^2}^{x} F \, dz \, dA(x,y)$$

Taking $F \equiv 1$, the volume of Ω is

$$|\Omega| = \iiint_{\Omega} 1 = \iint_{D} (x - x^2 - y^2) \, dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\cos\theta} (r\cos\theta - r^2) \, r \, dr \, d\theta = \cdots \quad \#$$

e.g. 3. Let T be the tetrahedron formed by vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.



Express

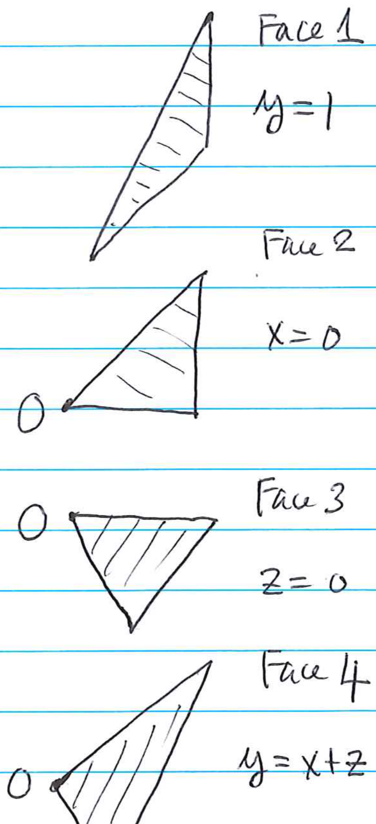
$$\iiint_T F$$

in

$$\iint_{\Delta_1} \int_{f_1}^{f_2} F \, dz \, dA(x,y),$$

$$\iint_{\Delta_2} \int_{f_1}^{f_2} F \, dx \, dA(y,z), \quad \text{and}$$

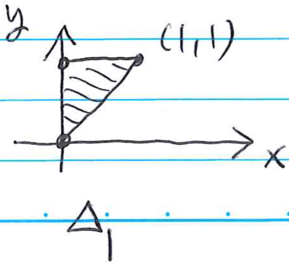
$$\iint_{\Delta_3} \int_{f_1}^{f_2} F \, dy \, d(x,z).$$



The tetrahedron projects along z-axis to Δ_1 on the x-y plane. Δ_1 has vertices $(0,0,0)$, $(1,1,0)$ and $(0,1,0)$. As a figure in the plane, its vertices are $(0,0)$, $(1,1)$, $(0,1)$ (exclude the third component)

$f_1 \equiv 0$, $f_2(x,y) = y - x$

$$\therefore \iiint_T F = \iint_{\Delta_1} \int_0^{y-x} F \, dz \, dA(x,y)$$

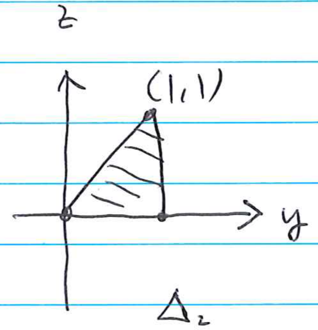


$$= \int_0^1 \int_0^y \left(\int_0^{y-x} F\, dz \right) dx\, dy .$$

T projects along x-axis onto y-z plane, Δ_2 has vertices

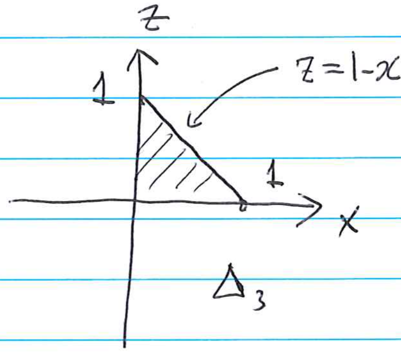
$(0,0,0)$, $(0,1,0)$, $(0,1,1)$, that's, $(0,0)$, $(1,0)$, $(1,1)$ in y-z plane.
$f_1 = 0$, $f_2(y,z) = y-z$

$$\iiint_T F = \iint_{\Delta_2} \int_0^{y-z} F\, dx\, dA(y,z)$$



$$= \int_0^1 \int_0^y \int_0^{y-z} F\, dx\, dz\, dy .$$

Finally, Δ_3 $(1,1,0)$, $(0,1,0)$, $(0,1,1)$
and $(1,0)$, $(0,0)$, $(0,1)$ after
projected onto x-z plane.



$f_1(x,z) = x+z$
$f_2(x,z) \equiv 1$

$$\iiint_T F \quad \iint_{\Delta_3} \int_{x+z}^{1} F\, dy\, dx\, dz$$

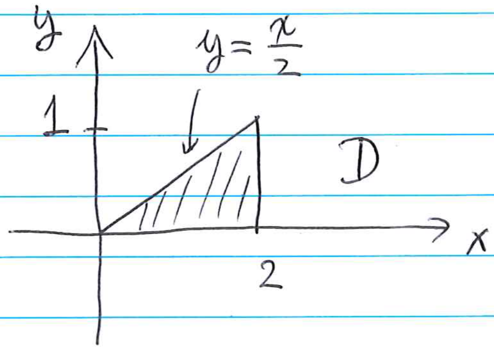
$$= \int_0^1 \int_0^{1-x} \int_{x+z}^{1} F\, dy\, dz\, dx .$$

e.g. 4 Evaluate

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{\cos x^2}{2\sqrt{z}}\,dx\,dy\,dz.$$

This is not a triple $\int$ but 3 iterated single integral.

$\int_0^1 \int_{2y}^2 \cos x^2\,dx\,dy$ not easy to do, so change the order of integration



$D = \{(x,y) : 0 \le y \le 1,\ 2y \le x \le 2\}$

$$\int_0^1 \int_{2y}^2 \cos x^2\,dx\,dy$$

$$= \iint_D \cos x^2\,dA$$

$$= \int_0^2 \int_0^{\frac{x}{2}} \cos x^2\,dy\,dx$$

$$= \int_0^2 \frac{x}{2}\cos x^2\,dx = \frac{1}{4}\int_0^4 \cos t\,dt = \frac{1}{4}\sin 4.$$

$$\therefore \int_0^4 \int_0^1 \int_{2y}^2 \frac{\cos x^2}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^4 \frac{1}{2\sqrt{z}} \int_0^1 \int_{2y}^2 \cos x^2\,dx\,dy$$

$$= \frac{1}{8}\sin 4 \int_0^4 \frac{1}{\sqrt{z}}\,dz \quad \text{(an improper integ)}$$

$$= \frac{1}{8}\sin 4 \times 2z^{\frac{1}{2}}\Big|_0^4 = \frac{1}{2}\sin 4 \quad \#$$

Another form of Fubini theorem.

For $\Omega \subseteq \mathbb{R}^3$, let

$$\Omega_z = \{(x,y) : (x,y,z) \in \Omega\}$$

This is the cross section of Ω at level z. Let

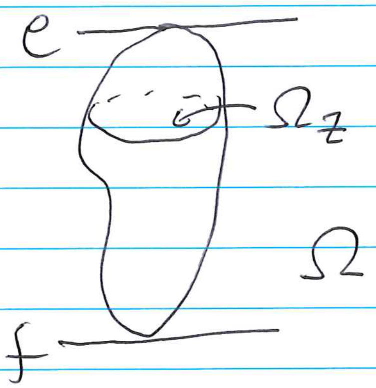
$|\Omega_z|$ be its area, while

$|\Omega|$ the volume of Ω.

<u>Theorem 3</u> Let F be continuous in Ω. Then

$$\iiint_\Omega F = \int_e^f \iint_{\Omega_z} F(x,y,z)\, dA(x,y)\, dz$$

where Ω is tightly bdd by $z = e, f$.



"Pf" Let $\Omega \subset B$, F extended to $\tilde{F}$.

$$\iiint_\Omega F = \iiint_B \tilde{F}$$

$$\sim \sum_{i,j,k} \tilde{F}(x_i^*, y_j^*, z_j^*)\, \Delta x_i\, \Delta y_j\, \Delta z_j$$

$$= \sum_k \left(\sum_{i,j} \tilde{F}(\quad)\, \Delta x_i\, \Delta y_j \right) \Delta z_j$$

$$\sim \sum_k \iint_{B_z} \tilde{F}(x, y, z_k^*)\, dx\,dy \,\Delta z_k$$

B_z cross section of B

$$= \sum_k \iint_{\Omega_z} F(x, y, z_k^*)\, dx\,dy \,\Delta z_k$$

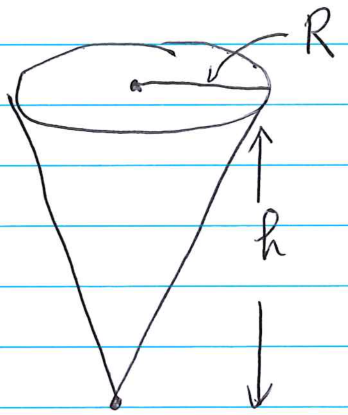
$$\sim \int_e^f \iint_{\Omega_z} F(x, y, z_k^*)\, dx\,dy \,\Delta z_k$$

$\left(\because \text{ for } z > f \text{ or } z < e,\ \Omega_z = \phi \text{ empty set, so } |\Omega_z| = 0 \right)$

$$\to \int_e^f \iint_{\Omega_z} F(x, y, z)\, dx\,dy \,dz . \quad \#$$

e.g. 5 Find the volum of the circula cone

$$z = \frac{h}{R}\sqrt{x^2+y^2}$$

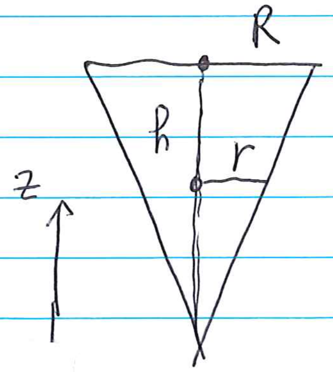


Let C be the cone

C_z = cross section at z

$|C_z| = \pi r^2$,

To find r use proportion



$$\frac{r}{R} = \frac{z}{h} \quad \therefore r = \frac{R}{h} z$$

$$|C_z| = \pi \left(\frac{R}{h} z\right)^2$$

By thm 3,

$$|C| = \int_0^h |C_z|\,dz$$

$$= \int_0^h \pi\left(\frac{R}{h}z\right)^2 dz$$

$$= \pi\frac{R^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi R^2 h . \#$$

A corollary of thm 3 is the Cavalieri's Principle:

If two objects have equal area cross sections, then their volume are the same.

It's clear from

$$|\Omega| = \int_e^f |\Omega_z|\,dz .$$